Collisionless Plasma Oscillations, Their Residual Presentation and the Mechanism of Landau Damping

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The description of the electron oscillations of a collisionless plasma by the usual residual presentation is insufficient in the initial phase of time development and for perturbations of small velocity spread. We derived criteria for the number of residual terms which have to be taken into account and obtain analytic expressions for the remaining integral. Decomposing the initial perturbation into velocity beams we show that Landau damping is due to phase mixing caused by free streaming of particle beams modified through the response of the main plasma body.

I. Introduction

Vlasov [1] derived the dispersion relation for longitudinal electrostatic waves in a collisionless plasma. Landau [2] extended his result showing that an arbitrary (analytic) perturbation of the collisionless plasma propagates approximately in the form of longitudinal waves which appear exponentially damped in time.

Wave damping in a system free of dissipative effects naturally raised the interest of experimental and theoretical physicists. For some time even the existence of Landau damping was questioned [3].

Landau's work was criticized for the validity of the linearization (Backus [4]) and of the time asymptotic representation of the solution (Hayes [5], Weitzner [6]) and for the use of the dispersion model underlying Vlasov's equation (Ecker and Hölling [7]).

Since Landau's derivation was a purely mathematical one and did not show any physical mechanism several papers try to provide a physical explanation of Landau damping (Jackson [8], Dawson [9]).

Recently the existence of Landau damping has been established experimentally (Malmberg et al. [10]) and its non dissipative nature was confirmed by the observation of echoes (Malmberg et al. [11]).

II. The Phenomenon of Landau Damping

Consider a fully ionized collisionless unbounded plasma in which the ions, merely providing a homogeneous smeared out neutralizing charge back ground, are regarded as immobile and where the particles interact through a curlfree electric field.

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In this model the time development of a sufficiently small perturbation $f(k,u,t)$ of a homogeneous and stationary electron distribution $F(u)$ is given by the linearized Vlasov equation

$$\left(\frac{\partial}{\partial t}+i\,k\,u\right)f(k,u,t)=i\,k\,\frac{\omega_p^2}{k^2}\,F'(u)\,n(k,t)\,,$$

$$n(k,t)=\int_{-\infty}^{+\infty}f(k,u,t)\,\mathrm{d}u\,. \tag{1}$$

The initial value problem of Eq. (1) can be solved by means of Laplace transformation (Landau [2]) or by expanding a given initial perturbation $f(k,u,t=0)$ in a sum of eigenfunctionals (Van Kampen [12], Case [13]).

We use the notations of the Van Kampen formalism: For any square integrable function $h(x)$ with the Fourier transform $h(p)$ there are defined positive and negative frequency parts by

$$h_+(z)=\int_0^{\infty}h(p)\,e^{ipz}\,\mathrm{d}p\,,\quad h_-(z)=\int_{-\infty}^{0}h(p)\,e^{ipz}\,\mathrm{d}p \tag{2}$$

which are analytic in the upper ($\mathscr{C}_+$) or lower ($\mathscr{C}_-$) complex half plane respectively. On the real axis $h(x)=h_+(x)+h_-(x)$ holds. In terms of these plus functions the time dependent density perturbation is given for a stable * stationary state $F(u)$ and for positive times in the form

$$n(k,t)=\int_{-\infty}^{+\infty}\frac{f_+(k,\omega/k,t=0)}{\varepsilon_+(k,\omega/k)}\,e^{-i\omega t}\,\mathrm{d}(\omega/k)\,;\quad t>0\,. \tag{3}$$

* By stability is meant stability according to Penrose's criterion (Penrose [14]) which guarantees the absence of eigenmodes of complex phase velocity and consequently the completeness of the real spectrum Van Kampen modes.

The denominator of this integral is the positive frequency part of the dielectric function

$$\varepsilon(k, \omega/k) = 1 - \frac{\omega_p^2}{k^2} \int_{-\infty}^{+\infty} \frac{F'(u)}{u - \omega/k} \, du \,. \qquad (4)$$

The integral (3) decays by phase-mixing. An asymptotic evaluation for large times t is possible if the distribution functions $f(z)$ and $F(z)$ can be continued analytically into some strip $0 \geqq \operatorname{Im}(z) \geqq -\alpha$. In this case $f_+(z)$ and $\varepsilon_+(z)$ have analytic continuations in the same domain. By application of the theorem of residues we get then (for $t > 0$)

$$n(k,t) = \sum_j \operatorname*{Res}_{\omega=\omega_j(k)} n(k,\omega)\, e^{-i\omega_j t} + \int_\Gamma n(k,\omega)\, e^{-i\omega t} \frac{d\omega}{2\pi i} \qquad (5)$$

with

$$n(k,\omega) = \frac{2\pi i}{k} \frac{f_+(k, \omega/k, t=0)}{\varepsilon_+(k, \omega/k)} . \qquad (6)$$

The sum is extended over all residues lying above the contour Γ shifted arbitrarily within the domain of analyticity.

If the integral term of Eq. (5) can be neglected it follows that in the limit $t \to \infty$ the contribution of the zero of ε_+ situated closest to the real ω-axis is dominant. The time development of the perturbation becomes asymptotically an exponentially damped harmonic oscillation. This damping process is called Landau damping.

III. Statement of the Problem and Intent

The expansion (5) gives the exact description of the time dependence of our problem. However, it is common practice to restrict the evaluation and discussion of this time dependence to the consideration of the first terms.

It is our intent to investigate whether such a procedure is justified for reasonable times. To this end we proceed as follows:

First – to justify the above practice – we should give conditions under which the contribution of the integral can be neglected in comparison to the sum of the residuals. In the range outside of these conditions we aim to develop an analytic expression for the finite contribution of the integral (Section IV. 1).

Secondly – in the range where the integral contribution is negligible – we investigate the convergence of the sum of the residuals and determine how many terms should be taken into account to represent the sum with a given accuracy (Section IV. 2).

Thirdly we study the circumstances under which the usual representation of the sum by the "weakest damped Landau pair of residuals" is a good approximation (Section IV. 3).

Finally the preceding investigation of the various contributions to the time density development sheds light on the physical background of the analytic description which we discuss in Section V.

The appendices provide information on the dielectric function and its zeros which is needed for the above described analysis.

Our results are illustrated by numerical evaluations the procedures of which are also described in the appendices.

IV. Criticism of the Asymptotic Evaluation of the Density Perturbation

To be specific we now restrict our investigation to the most important case of Maxwellian distributions for the unperturbed plasma and the initial perturbation:

$$F(u) = [1/\sqrt{2\pi}\, u_0] \exp\{-u^2/2\, u_0^2\} , \qquad (7)$$

$$f(k, u, t=0) = A(k)\, [\beta/\sqrt{2\pi}\, u_0] \exp\{-\beta^2 u^2/2\, u_0^2\} . \qquad (8)$$

Note that due to the parameter β the widths of the unperturbed and perturbed distributions are different.

IV. 1 Contribution of the Remaining Integral

By the choice (7), (8) for the velocity distribution functions we ensure that f_+ and ε_+ can be continuated analytically into the entire complex ω/k plane. Therefore the remaining integral of Eq. (5)

$$\int_{\Gamma_n} d\nu\, e^{-ik\nu t} \frac{f_+(k,\nu,t=0)}{\varepsilon_+(k,\nu)} = \frac{(k\lambda_D)^2}{2\pi i} \int_{\Gamma_n} d\zeta\, I(\zeta)\, e^{-i\sqrt{2}\, k u_0 \zeta t} , \qquad (9)$$

where

$$I(\zeta) = \frac{Z(\beta\zeta)}{1 + (k\lambda_D)^2 + \zeta Z(\zeta)} ,$$

can be studied for a sequence of integration paths with the following properties *:

a) Γ_n avoids the zeros of ε_+ within $\mathscr{C}_-$

b) The index n numerates the number of pairs of residues included in the sum of residues

c) For $n \to \infty$ the distance of Γ_n from the origin tends to infinity

We choose half-circles

$$\Gamma_n = \{\zeta \mid \zeta = \sigma_n e^{i\varphi}, 0 \geqq \varphi \geqq -\pi\}$$

for the integration contour in order to make use of Jordan's lemma **. It is sufficient to discuss the domain $0 \geqq \varphi \geqq -\pi/2$ since the integral along $-\pi/2 \geqq \varphi \geqq -\pi$ is the complex conjugate of the integral along the arc $0 \geqq \varphi \geqq -\pi/2$ which follows from the symmetry relation $Z(-\zeta^*) = -Z^*(\zeta)$.

Choosing σ_n so large that the asymptotic representation of the dispersion function

$$Z(\zeta) \underset{\substack{\zeta \to \infty \\ \operatorname{Im} \zeta < 0}}{\sim} 2\,i\sqrt{\pi} \exp\{-\zeta^2\} - 1/\zeta \tag{10}$$

is valid along Γ_n we get for $I(\zeta)$ the approximation

$$I = \frac{Z(\beta\,\zeta)}{1 + (k\,\lambda_{\mathrm{D}})^2 + \zeta\,Z(\zeta)} \sim \frac{2\,i\sqrt{\pi}\exp(-\beta^2\,\zeta^2) - (\beta\,\zeta)^{-1}}{(k\,\lambda_{\mathrm{D}})^2 + 2\,i\sqrt{\pi}\,\zeta\exp(-\zeta^2)}\,. \tag{11}$$

For $|\zeta| \gg (k\,\lambda_{\mathrm{D}})^2/2\sqrt{\pi}$ the poles of the integrand are situated close to the ray $\varphi = -\pi/4$ (or $\varphi = -3/4\,\pi$) [cf. Equation (A 2)]. The integration path avoids these singularities if we ensure for $\varphi = -\pi/4$

$$\exp\{i(\varphi - \sigma^2 \sin 2\,\varphi + \pi/2)\} = 1$$

or

$$\sigma_n = \sqrt{\pi(2\,n - 1/4)}\,. \tag{12}$$

For $n \to \infty$ the distance of the integration path from the origin thus tends to infinity.

Distinguishing three intervals of the azimuthal angle φ

(a) $0 \geqq \varphi \geqq -\dfrac{\pi}{4} + \dfrac{1}{2\,\sigma_n^2} \ln \dfrac{4\sqrt{\pi}\,\sigma_n}{(k\,\lambda_{\mathrm{D}})^2}$,

(b) $-\dfrac{\pi}{4} + \dfrac{1}{2\,\sigma_n^2} \ln \dfrac{4\sqrt{\pi}\,\sigma_n}{(k\,\lambda_{\mathrm{D}})^2} \geqq \varphi \geqq -\dfrac{\pi}{4}$,

and (c) $-\pi/4 \geqq \varphi \geqq -\pi/2$

we get the following estimates:

* $\zeta = \omega/\sqrt{2}\,k\,u$/ For $Z(\zeta)$ see Appendix A.

** The integrals along the real axis from $-\infty$ to $-\sigma_n$ and from σ_n to $+\infty$ do not contribute in the limit $\sigma_n \to \infty$, since $I(\zeta) \sim 1/\zeta$ holds for large $|\zeta|$ on the real axis (compare the following estimate of I in the domain of small azimuthal angle which is still valid on the real axis).

In the domain (a)

$$|(k\,\lambda_{\mathrm{D}})^2 + 2\,i\sqrt{\pi}\,\zeta\,e^{-\zeta^2}| > \tfrac{1}{2}(k\,\lambda_{\mathrm{D}})^2$$

and consequently

$$I \sim \frac{2}{\beta(k\,\lambda_{\mathrm{D}})^2}\,\frac{1}{\zeta} \quad \text{for} \quad |\zeta| \to \infty$$

holds.

In the domain (c) we get

$$I \sim \frac{1}{\zeta}\exp\{-\zeta^2(\beta^2 - 1)\}\,.$$

If $\beta \leqq 1$ holds then I has the asymptotic behaviour $1/\zeta$ for $|\zeta| \to \infty$ in this domain, too.

In the interval (b) the integrand becomes large because the integration path comes close to zeros of the dispersion function. We have choosen σ_n thus that $i\,\zeta\exp(-\zeta^2)$ takes for $\varphi = -\pi/4$ a real and positive value [see Equation (12)]. The increment $\Delta\varphi$ necessary to make

$$\exp\{+i\,\varphi - i\,\sigma^2 \sin 2\,\varphi + i\,\pi/2\}$$

purely imaginary is determined by

$$+\Delta\varphi - 2\,\sigma^2(\Delta\varphi)^2 = \pm\,\pi/2$$

or since $\quad \Delta\varphi = o\left(\dfrac{1}{\sigma^2}\ln\left(\dfrac{(k\,\lambda_{\mathrm{D}})^2}{\sigma}\right)\right) \quad$ holds

by $\quad |\Delta\varphi| \cong \dfrac{\sqrt{\pi}}{2}\,\dfrac{1}{\sigma}$.

For

$$-\frac{\pi}{4} + \frac{\sqrt{\pi}}{2}\,\frac{1}{\sigma} \geqq \varphi \geqq -\frac{\pi}{4}$$

the estimate $|(k\,\lambda_{\mathrm{D}})^2 + 2\,i\sqrt{\pi}\,\zeta\,e^{-\zeta^2}| > (k\,\lambda_{\mathrm{D}})^2$ holds. This interval covers the domain (b) completely for $\sigma \to \infty$. Consequently the integrand I behaves like $1/\zeta$ in the domain (b) when $|\zeta| = \sigma$ tends to infinity.

Summarizing, we have shown that

1. for $\beta \leqq 1$ $I(\zeta)$ behaves like $1/\zeta$ on the whole integration contour for $|\zeta| \to \infty$. According to Jordan's lemma the integral (9) vanishes for $n \to \infty$, the time t being positive nonvanishing,

2. for $\beta > 1$ in the interval (c) $I(\zeta)$ tends to infinity like $\zeta^{-1}\exp\{-\zeta^2(\beta^2 - 1)\}$ for $|\zeta| \to \infty$. In this case the integral (9) gives a contribution which must be added to the sum of residues.

We now estimate this contribution which is not covered by the sum of residues when $\beta > 1$ holds.

We do this by constructing a function $h(\zeta)$ holomorphic in $\mathscr{C}_- - \{0\}$ with the properties

$$\frac{Z(\beta\zeta)}{1+(k\lambda_{\mathrm{D}})^2+\zeta Z(\zeta)} - h(\zeta)\to 0 \quad \text{for} \begin{cases} |\zeta|\to\infty\,, \\ \zeta\in\mathscr{C}_- - \{0\}\,; \end{cases}$$

$h(\zeta)\to 0$ for $|\zeta|\to\infty$ within the intervals (a) and (b)

$$Z(\beta\zeta) - h(\zeta)\,[1+(k\lambda_{\mathrm{D}})^2+\zeta Z(\zeta)] \sim e^{-\zeta^2}(\beta^2-[\beta^2]) \quad \text{for } |\zeta|\to\infty \text{ within (c)}.$$

$[\beta^2]$ denotes the largest integer smaller than β^2. The contribution not met by the sum of residues then is given by

$$n_I(t) = \frac{(k\lambda_{\mathrm{D}})^2}{2\pi i} \int_{\mathscr{R}-i0} h(\zeta)\, e^{-\sqrt{2}\,k u_0 \zeta t}\,\mathrm{d}\zeta\,. \qquad (13)$$

The construction of $h(\zeta)$ and the evaluation of the integral (13) is straightforward but somewhat lengthy. We therefore just give the result:

In the range $2\geqq\beta^2>1$ we find explicitely

$$n_I(t) = \frac{(k\lambda_{\mathrm{D}})^2}{2}\,\mathrm{erfc}\left(\frac{k u_0 t}{\sqrt{2(\beta^2-1)}}\right); \quad 2\geqq\beta^2>1\,. \qquad (14)$$

For $\beta^2>2$ we can give only an approximation

$$n_I(t) = \frac{(k\lambda_{\mathrm{D}})^2}{2}\sum_{\nu=1}^{[\beta^2]}(-k^2\lambda_{\mathrm{D}}^2)^{\nu-1}\left(\frac{\beta^2-\nu}{\pi}\right)^{\nu-1} \cdot i^{\nu-1}\,\mathrm{erfc}\left(\frac{k u_0 t}{\sqrt{2(\beta^2-\nu)}}\right). \qquad (15)$$

Here, $i^\nu\,\mathrm{erfc}(x)$ denotes the repeated integral of the error function.

Numerical evaluations are shown in Figs. 1 and 2 *. Figure 1 demonstrates that for equal temperatures of the perturbation and the unperturbed plasma the representation by sums of residues is valid after one plasma period. If the velocity distribution of the perturbation has a smaller width, however, the exact result differs from the sum of residues for the entire time interval in which there still exists an appreciable perturbation (Figure 2). This discrepancy is due to the nonvanishing remaining integral. The analytic expression (14) was verified numerically.

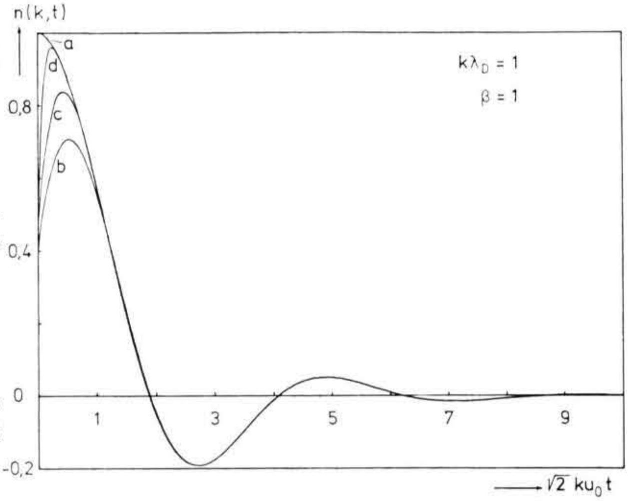

Fig. 1. Exact perturbation development (a) compared with approximations by sums of one (b), three (c), and twenty-five (d) pairs of residues.

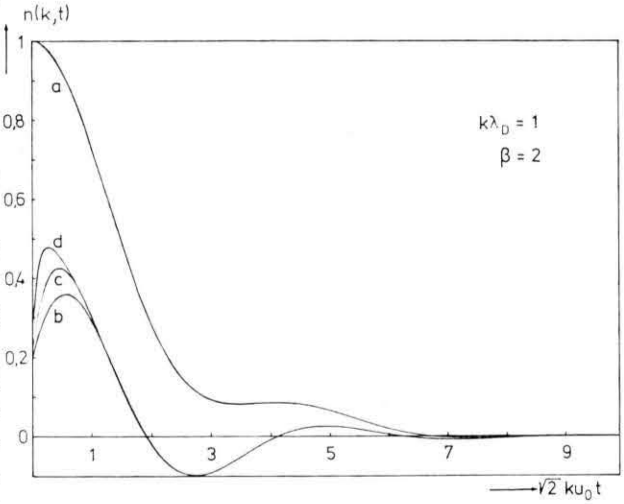

Fig. 2. Time development of a perturbation of a smaller width (a). For comparison the sums of 1, 3, resp. 25 pairs of residues [(b) to (d)] are given.

IV. 2 Convergence of the Sum of Residues

We consider the convergence of the sum of residues for the most important case $\beta=1$ only. Using the theorem of residues for the evaluation of the density perturbation we get

$$n_{\mathrm{Res}}(k,t) = -(k\lambda_{\mathrm{D}})^2\sum_n \exp\{-i\sqrt{2}\,\zeta_n k u_0 t\}\,Z(\zeta_n)/[\zeta_n Z'(\zeta_n)+Z(\zeta_n)]. \qquad (16)$$

* The numerical procedure used is described in Appendix D.

By ζ_n we denote the series of zeros of the dispersion equation. Inserting

$$Z(\zeta_n) = -\frac{1}{\zeta_n}[1+(k\lambda_D)^2], \quad Z'(\zeta_n) = -2[1+\zeta_n Z(\zeta_n)] = 2(k\lambda_D)^2, \tag{17}$$

using the notation $\zeta_n = x_n + i y_n$ and collecting the residues at the poles ζ_n and $-\zeta_n^*$ yields

$$\frac{n_{\text{Res}}(k,t)}{2[1+(k\lambda_D)^2]} = \sum_n \frac{r_n\cos(\sqrt{2}\,x_n k u_0 t) - s_n \sin(\sqrt{2}\,x_n k u_0 t)}{r_n^2+s_n^2}\exp\{\sqrt{2}\,y_n k u_0 t\}, \tag{18}$$

$$r_n := 2x_n^2 - 2y_n^2 - 1 - (k\lambda_D)^2; \quad s_n := 4x_n y_n .$$

To check the convergence of this series we consider its terms for N so large that the asymptotic formulas (A 2) hold for $n \geqq N$. Using

$$x_n^2 - y_n^2 \cong \ln\left(\varrho_n \frac{2\sqrt{\pi}}{(k\lambda_D)^2}\right); \quad x_n y_n \cong -(n+\tfrac{3}{8})\pi \tag{19}$$

and neglecting $\ln[2\pi/(k\lambda_D)^2] - 1 - (k\lambda_D)^{-2}$ against $\ln(2n+\frac{3}{4})$ the remainder after N terms becomes

$$\frac{|R_N(k,t)|}{2[1+(k\lambda_D)^2]} \simeq \sum_{n=N}^{\infty} e^{-\varkappa_n}\frac{\ln(2n+\frac{3}{4})\cos\varkappa_n + (4n+\frac{3}{2})\sin\varkappa_n}{[\ln(2n+\frac{3}{4})]^2+\pi^2(4n+\frac{3}{2})^2}, \quad \varkappa_n := \sqrt{2\pi(n+\tfrac{3}{8})}\,k u_0 t. \tag{20}$$

Estimating the terms of the series (20) by

$$\left| e^{-\varkappa_n}\frac{\ln(2n+\frac{3}{4})\cos\varkappa_n + (4n+\frac{3}{2})\sin\varkappa_n}{[\ln(2n+\frac{3}{4})]^2+\pi^2(4n+\frac{3}{2})^2}\right| < \frac{\ln(2n+\frac{3}{4}) + (4n+\frac{3}{2})e^{-\varkappa_n}(1-\delta_{t,0})}{\pi^2(4n+\frac{3}{2})^2} \tag{21}$$

the absolute convergence of the sum of residues follows for $0 \leqq t \leqq \infty$:

$$\frac{|R_N(k,t)|}{2[1+(k\lambda_D)^2]} < \left\{\frac{\ln(2N+\frac{3}{4})+1}{4\pi^2(2N+\frac{3}{4})} + (1-\delta_{t,0})\,e^{-\varkappa_N}\ln\left(1+\frac{1}{\varkappa_N}\right)\right\}. \tag{22}$$

For $t \to 0$ the series of moduli does not converge uniformly. Improving the estimate of the sum over the sine terms (which give the nonuniformity at $t=0$)

$$\sum_{n=N}^{\infty}\frac{\exp(-\varkappa_n)\sin\varkappa_n}{4\pi(n+3/8)} \approx \tfrac{1}{2}\,\text{Im}\,E_1[\varkappa_n(1-i)] \tag{23}$$

we see that the value of R_N is still time dependent through $\varkappa_N = [2\pi(N+3/8)]^{1/2} k u_0 t$. In order to guarantee a given degree of exactness of the representation by the sum of residues N must be increased for $t \to 0$ according to

$$N \sim (k u_0 t)^{-2}. \tag{24}$$

The nonuniformity of convergence in the initial phase – which can bee seen in Figs. 1 and 2 already – is demonstrated in Fig. 3 by a drastic increase in the number of residues summed up.

IV. 3 Validity of the One Pole Approximation

The magnitude of the n-th residue is given by the exponential $\exp\{\sqrt{2}\,y_n k u_0 t\}$. To ensure that the relative error of the representation by the least

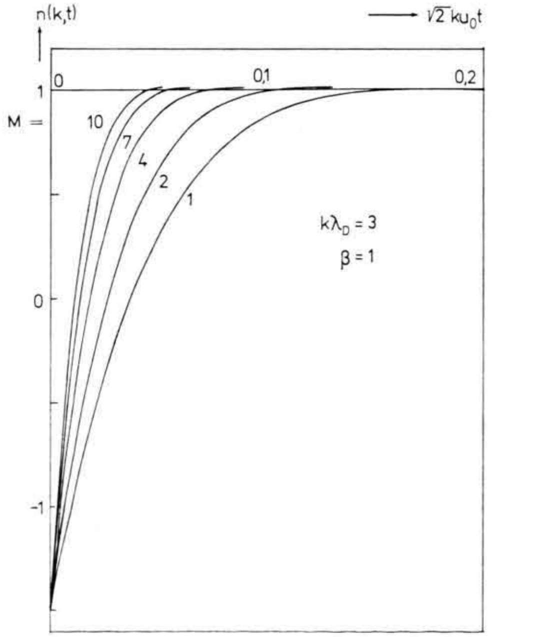

Fig. 3. Nonuniformity of convergence of the sum of residues at $t=0$. The sum of residues extends over $M \cdot 100$ pairs of residues.

damped pair of residues alone is smaller than ε it is necessary to demand

$$\exp\{\sqrt{2}\, k\, u_0\, t(y_1 - y_0)\} < \varepsilon . \tag{25}$$

Our numerical evaluations show that after one period of oscillation the further residues do not contribute essentially. From Fig. 4 we see that for the rather modest exactness requirement $\varepsilon = 0.1$, already the condition $t \gtrsim T(k)$ must be fulfilled. Consequently it is not restrictive to neglect the residues $n \geqq 2$ for the discussion of the validity of the one pole representation.

Since $y_1 < y_0 < 0$ holds there always exists a time fulfilling condition (25). In Fig. 4 we show this minimum time in units of the oscillation period $T(k) = \sqrt{2}\,\pi / k\, u_0\, x_0$.

We remark that the minimum time of Fig. 4 for the validity of the representation by one residue holds for the exact values of the least damped branch of the dispersion equation, only. The validity of Landau's formulas (A 3, A 4) is further restricted by the asymptotic method of evaluation to the domain $k\lambda_D \lesssim 0.3$.

V. Physical Mechanism of Landau Damping

Free-motion-models for Landau damping were already suggested in the pioneering time of Vlasov theory. In a system of non-interacting particles the velocity spread causes heavy damping of spatial inhomogeneities proportional to $\exp(-\alpha t^2)$. Collective electrostatic forces tend to maintain order with the result that perturbations in such a long range interacting system disappear less rapidly according to $\exp(-\gamma t)$. The original ideas of the "FREE STREAMING MODEL" or "PHASE MIXING MODEL" have been elucidated by Van Kampen [15].

The main criticism of the above models is that so far one has not been able to move from a merely qualitative description to a quantitative account.

Other attempts to interprete Landau damping may be grouped under the heading "ENERGY EXCHANGE MODELS". The fact that a system with a particle velocity distribution which exhibits a gap around the phase velocity of the wave does not show Landau damping seems to indicate that the Landau damping phenomenon is closely related to those particles which move with velocities near the phase

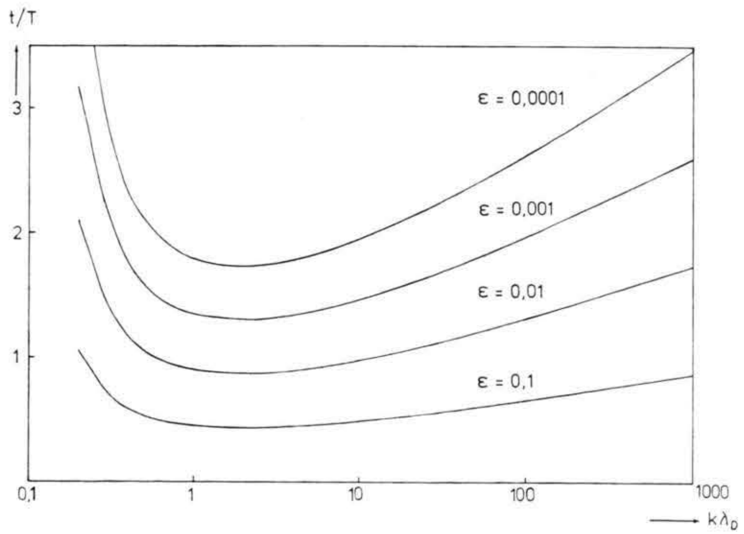

Fig. 4. Minimum time (in units of the oscillation period) after which the representation by the Landau poles reaches the relative exactness ε.

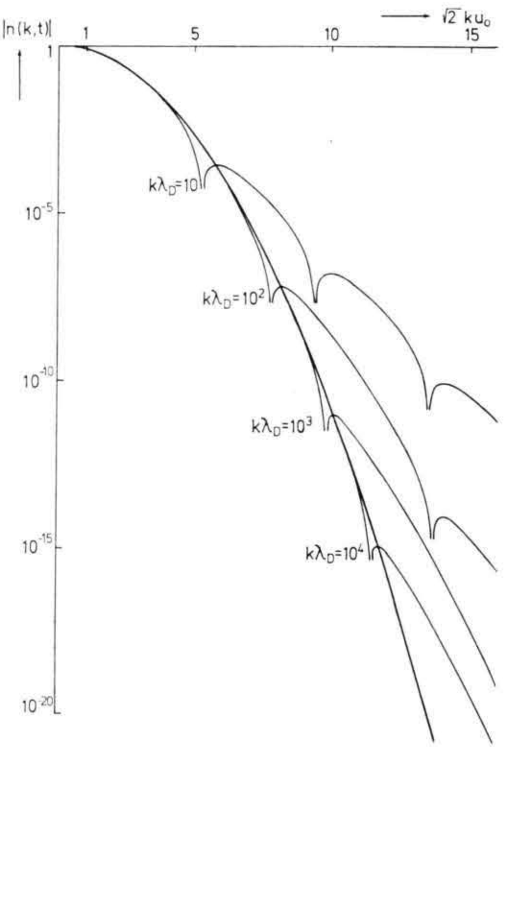

Fig. 5. Time development of the absolute value of the density $|n(k,t)|$ for various values of $k\lambda_D$ als calculated from the Vlasov equation (light lines) and as calculated for free streaming (heavy line). Note that the tips (tending to $-\infty$) indicate the zeros of the oscillation.

velocity. The term "energy exchange models" embraces the "TRAPPED PARTICLE MODEL [8]" and the "RESONANT ENERGY TRANSFER MODEL [9, 16]".

Both "energy exchange models" allow in the range of long wave lengths the quantitative calculation of the Landau damping decrement, – apart from a numerical factor. However the basically macroscopic approach of these models clearly is unable to give any information about microscopic details of the Landau mechanism.

The "trapped particle model" suffers from the fact that trapping and untrapping of particles are non-linear effects and cannot be responsible for Landau damping which is a linear effect.

The "resonant energy transfer model" does not clearly define the interval of the resonant velocities. Furthermore the assumption appears not very convincing that at the time $t=0$ the resonant particles are neither affected by nor do affect the electrostatic wave.

We aim here to corroborate the "phase mixing model" by developing it to a quantitative formulation.

Let us first compare the time development of the density of Vlasov solutions with that of the corresponding interactionfree free streaming model.

The corresponding results – numerical solutions – are found in Figure 5.

The figure shows that in the initial phase the free streaming decay is always a good approximation.

The effect of the collective interaction on the free streaming decay is shown by the deviation of the Vlasov curves from the free streaming curve to higher values. It occurs at a later time if $k\lambda_D$ is larger. Even this can be understod if one remembers that collective influences develop on a time scale ω_p^{-1} which in the units $(k u_0)^{-1}$ (characteristic free streaming time) turns out to be proportional to $k\lambda_D$.

Damping of an initial velocity perturbation in a system of non-interacting particles appears in the form

$$n_f(k,t) = \int f(k,u,t=0)\, e^{-ikut}\, du = \int f_+\, e^{-ikut}\, du \quad \text{for} \quad t>0. \tag{26}$$

When the particles do interact the different velocities u contribute to a different amount due to the response function $\varepsilon_+(k,u)$:

$$n(k,t) = \int du \frac{f_+(k,u,t=0)}{\varepsilon_+(k,u)} e^{-ikut} \quad \text{for} \quad t>0. \tag{27}$$

The density develops in time as if we had free streaming with an initial perturbation modified by the factor ε_+^{-1}. Therefore the problem is to understand physically the modification of the contribution of each velocity beam by the plasma response.

To this end we study the plasma response in a form similar to the test particle picture developed by Rostoker [17]. Consider as an initial perturbation a beam of electrons of velocity u_b.

$$f_b(u,t=0) = \delta(u-u_b). \tag{28}$$

Vlasov's equation

$$i k[u-(\omega/k)-i0]\, f_b(k,u,\omega) = i(\omega_p^2/k)\, F'(u) \int du'\, f_b(k,u',\omega) + f_b(t=0) \tag{29}$$

gives for the Fourier transform of the perturbation

$$f_b(k,u,\omega) = \left\{ \frac{\omega_p^2}{k^2} \frac{F'(u)}{(u-\omega/k-i0)\,\varepsilon_+(k,\omega/k)} + \delta(u-u_b) \right\} \frac{1}{i k (u_b-\omega/k-i0)}. \tag{30}$$

The inverse transformation yields

$$f_b(k,u,t) = e^{-iku_bt} \int d\left(\frac{\omega}{k}\right) \frac{\exp\{i k(u_b-\omega/k)t\}}{2\pi i(u_b-\omega/k-i0)} \left\{ \frac{\omega_p^2}{k^2} \frac{F'(u)}{(u-\omega/k-i0)\,\varepsilon_+(\omega/k)} + \delta(u-u_b) \right\}. \tag{31}$$

For large times the factor $\exp\{ik(u_b-\omega/k)t\}/2\pi i(u_b-\omega/k-i0)$ can be replaced by the Dirac function $\delta(u_b-\omega/k)$. Therefore we write

$$f_b(k,u,t) = e^{-iku_bt} \frac{1}{\varepsilon_+(u_b)} \tilde{f}_{u_b}(u) + \Delta_{u_b} \tag{32}$$

where

$$\tilde{f}_{u_b}(u) = \frac{\omega_p^2}{k^2} \frac{F'(u)}{u-u_b-i0} + \varepsilon_+(u_b)\,\delta(u-u_b) = \frac{\omega_p^2}{k^2} \mathcal{P} \frac{F'(u)}{u-u_b} + \left(1 - \frac{\omega_p^2}{k^2} \int \mathcal{P} \frac{F'(u)\,du}{u-u_b}\right) \delta(u-u_b) \tag{33}$$

is Van Kampen's mode of the phase velocity u_b and where Δ_{u_b} denotes the remainder

$$\Delta_{u_b}(u) = \int \mathrm{d}\left(\frac{\omega}{k}\right)\left[\frac{\exp\{-i\,k\,t(\omega/k)\}}{2\,\pi\,i(u_b-\omega/k-i\,0)} - e^{-iku_bt}\,\delta(u_b-\omega/k)\right]\left\{\frac{\tilde{f}_{\omega/k}(u)}{\varepsilon_+(\omega/k)} + \delta(u-u_b) - \delta(u-\omega/k)\right\}. \tag{34}$$

Averaging Δ_{u_b} with a test function $g(u_b)$ we find

$$\langle\Delta_{u_b}\rangle = \int \mathrm{d}\left(\frac{\omega}{k}\right)\frac{\tilde{f}_{\omega/k}(u)}{\varepsilon_+(\omega/k)}\int \mathrm{d}u_b\,g(u_b)\left[\frac{\exp\{-i\,k\,t(\omega/k)\}}{2\,\pi\,i(u_b-\omega/k-i\,0)} - e^{-iku_bt}\,\delta(u_b-\omega/k)\right]$$
$$+\int \mathrm{d}\left(\frac{\omega}{k}\right)\left[\frac{g(u)\exp\{-i\,k\,t(\omega/k)\}}{2\,\pi\,i(u-\omega/k-i\,0)} - \frac{g(\omega/k)\exp\{-i\,k\,t\,u\}}{2\,\pi\,i(\omega/k-u-i\,0)}\right]. \tag{35}$$

Since the integrated brackets disappear identically we have

$$\langle\Delta_{u_b}\rangle = 0 \tag{36}$$

for any test function. Note that Eq. (36) holds for all times $t > 0$.

Our calculation yields that the response of the plasma to the initial velocity beam together with that beam results in a Van Kampen mode modified with the factor ε_+^{-1} and an additional term Δ_{u_b} which does not contribute after averaging. Superposition of these dressed beams thus gives the general solution (27) of the initial value problem.

We have shown that Landau damping can be understood as a consequence of free streaming of particle velocity beams modified by the response of the plasma body. The damping is always weaker than that caused by corresponding free streaming without plasma response since the collective reactions counteract this latter dispersion effect. The non dissipative nature of Landau damping is self evident in this free streaming or phase mixing concept.

VI. Summary

We have shown that the evaluation by sums of residues of the plasma perturbation fails in the initial phase of time development and – which is of greater interest – for initial perturbations which are sharply peaked.

If the velocity spread of the perturbation is small in comparison to that of the unperturbed plasma then there remains a non negligible contribution of the contour integral. We give analytic results for this additional term.

If the velocity spread is equal or larger than that of the plasma body then the sum of residues converges towards the true perturbation function for positive times (non zero). However, to obtain a good approximation for small times the number of residues taken into account increases with t^{-2}.

After one oscillation all modes exept the two least damped ones have disappeared as can be seen from our numerical evalutions. The one pole-Landau-approximation is valid after a minimum time which depends of the wavelength as shown in Figure 4.

Our physical interpretation of Landau damping starts from a decomposition of the initial perturbation into velocity beams. Then we study the modification of these beams due to the response of the main plasma body and find Van Kampen modes multiplied by the factor ε_+^{-1} plus a remaining term. Applying the free streaming concept to these modified beams we arrive at the Landau solution. Landau damping is due to phase mixing caused by free streaming of particle beams modified through the response of the plasma body.

Appendices

A) Properties of the Dielectric Function

The positive frequency part of the dielectric function (4) takes for $F(u)$ Maxwellian the form

$$\varepsilon_+(\zeta) = 1 - \frac{\omega_p^2}{k^2 u_0^2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{t\,e^{-t^2}}{t-\zeta}\,\mathrm{d}t = 1 + \frac{\omega_p^2}{k^2 u_0^2}\left[1+\zeta Z(\zeta)\right] \tag{A 1}$$

when $\mathrm{Im}(\zeta) > 0$ holds. It can be defined by its analytic continuation when $\mathrm{Im}(\zeta) \leqq 0$ holds. ζ is the dimensionless phase velocity $\omega/\sqrt{2}\,k\,u_0$. $Z(\zeta)$ denotes the dispersion function which has been introduced by Jackson [8] and tabulated by Fried and Conte [19].

Since the Maxwellian distribution function is stable (cf. footnote * on page 1863) zeros of $\varepsilon_+(\zeta)$ can lie in $\mathscr{C}_-$ only where $\varepsilon_+(\zeta)$ is defined by analytic continuation.

$Z(\zeta)$ is holomorphic in the entire complex plane because it is the Cauchy integral of a function which is holomorphic in the entire plane. Since the function $e^{-\zeta^2}$ has an essential singularity at infinity the same holds for $Z(\zeta)$. Consequently $\varepsilon_+(\zeta)$ has an infinite number of zeros clustering at infinity.

From Eq. (A 1) we see that $-\zeta^*$ is a zero of ε_+ when ζ is a zero. Consequently we can restrict the discussion to $0 \geqq \varphi \geqq -\pi/2$ [we use the notation $\zeta = \varrho \exp(i\varphi)$].

For $\zeta \in \mathscr{C}_-$, $|\mathrm{Im}(\zeta)| \gtrsim |\mathrm{Re}(\zeta)| \gg 1$ an approximation of $Z(\zeta)$ can be justified which yields

$$\left.\begin{aligned} \varrho_n &= \sqrt{\pi(2n+3/4)} \\ \varphi_n &= -\frac{\pi}{4} + \frac{1}{2\varrho_n^2} \ln \frac{2\sqrt{\pi}\,\omega_p^2\,\varrho_n}{k^2 u_0^2} \end{aligned}\right\} \text{ for } n \gg 1 . \tag{A 2}$$

The zeros cluster at infinity along the ray $\varphi = -\pi/4$. In the limit $\zeta \in \mathscr{C}_-$, $|\mathrm{Im}(\zeta)| \ll |\mathrm{Re}(\zeta)|$, $|\zeta| \gg 1$ we get (consistently for $\omega_p^2/k^2 u_0^2 \gg 1$)

$$\zeta_r^2 = \frac{\omega_p^2}{2k^2 u_0^2} + \frac{3}{2} + 6\,\frac{k^2 u_0^2}{\omega_p^2} + \dots, \tag{A 3}$$

$$\zeta_i = -\sqrt{\pi}\,\zeta_r^4 \exp\{-\zeta_r^2\} \tag{A 4}$$

frequency and damping decrement of the branch of the least damped mode for $\lambda \gg \lambda_D$. Equation (A 4) gives the Landau damping decrement.

In the limit $\omega_p^2/k^2 u_0^2 \ll 1$ the modulus of the zeros gets large for the first branches of zeros already whereas the limit $\arg(\zeta) \cong -\pi/4$ still does not apply. In this case we get the system of equations

$$\left.\begin{aligned} k^2 u_0^2/2\sqrt{\pi}\,\omega_p^2 &= \sqrt{\zeta_r^2+\zeta_i^2}\exp\{\zeta_i^2-\zeta_r^2\} \\ \arctan(\zeta_i/\zeta_r) &= 2\,\zeta_i\,\zeta_r + \pi(2n+\tfrac{1}{2}) \end{aligned}\right\} \begin{aligned} &n = 0, 1, \dots, \\ &\omega_p^2/k^2 u_0^2 \ll 1 . \end{aligned} \tag{A 5}$$

By iterative solution of (A 5) another asymptotic representation can be obtained.

Especially for $n=0$ we get (with $|\zeta_i| \gg |\zeta_r|$)

$$-\zeta_i \exp\{\zeta_i^2\} = k^2 u_0^2/2\sqrt{\pi}\,\omega_p^2 \; ; \; \zeta_r = -\pi/2\,\zeta_i \tag{A 6}$$

the asymptotic expression for the branch of the least damped mode for $\omega_p^2/k^2 u_0^2 \ll 1$.

Apart from the asymptotic evaluations given the zeros of ε_+ are amenable to numeric calculation, only. We describe our numerical procedure for $Z(\zeta)$ and for the evaluation of the zeros of ε_+ in the following appendices.

In Fig. 6 we shown the first twenty-five branches of zeros of the dispersion equation. The imaginary part of the zeros is plotted against the real part of the zeros for $0 \leqq k\lambda_D \leqq 10^3$. To demonstrate the concentration of the zeros on the ray $-\mathrm{Im}(\zeta) = \mathrm{Re}(\zeta)$ we have drawn the curves $1 + \mathrm{Re}[\zeta Z(\zeta)] = -(k\lambda_D)^2$, which connect zeros of equal $k\lambda_D$ on different branches, too.

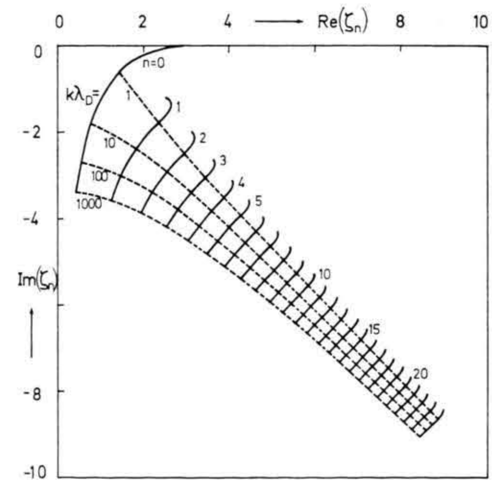

Fig. 6. The first twenty-five branches of zeros of the dispersion equation.

B) Numerical Procedures for the Dispersion Function

The dispersion function is closely related to the complex error function which we approximate by

$$\operatorname{erf}(x+i\,y) = \operatorname{erf} x + \frac{e^{-x^2}}{2\,\pi\,x}\,[(1-\cos 2\,x\,y) + i\sin 2\,x\,y] + \frac{2}{\pi}\,e^{-x^2}\sum_{n=1}^{\infty}\frac{\exp\{-\frac{1}{4}n^2\}}{n^2+4\,x^2}\,[f_n(x,y)+i\,g_n(x,y)] + \varepsilon(x,y)\,. \qquad \text{(B 1)}$$

Here the auxiliary functions f_n and g_n are given by

$$f_n = 2\,x - 2\,x\cosh n\,x\cos 2\,x\,y + n\sinh n\,y\sin 2\,x\,y\,,$$
$$g_n = 2\,x\cosh n\,x\sin 2\,x\,y + n\sinh n\,y\cos 2\,x\,y\,. \qquad \text{(B 2)}$$

The error $\varepsilon(x,y)$ of (B 1) can be estimated by

$$|\varepsilon(x,y)| \approx 10^{-16}\,|\operatorname{erf}(x+i\,y)|\,. \qquad \text{(B 3)}$$

This approximation is taken from the Handbook of Mathematical Functions [18].

For $\operatorname{Im}(\zeta)<0$ and $|\operatorname{Im}(\zeta)| \approx |\operatorname{Re}(\zeta)| \gg 1$ the dispersion function gets very steep due to the factor $\exp(-\zeta^2)$. This is the reason why our procedure works for $y<0$ in the domain $y^2-x^2<83$ only. If $y^2-x^2>10$ or $|y|>9$ or $|x|>9$ holds we rest content with an asymptotic representation.

Comparing our procedure with that of Fried and Conte [19] we can say that it is much simpler and more economical in the use of computer time. The numerical conformity with data given in [19] is almost complete.

C) Zeros of the Dispersion Equation

The zeros of the dispersion Eq. (A 1) are evaluated by Newton iteration. We start with the approximation

$$x = [1 + \tfrac{3}{2}\,(k\,\lambda_{\rm D})^2]\,(2\,k\,\lambda_{\rm D})^{-1}\,;\quad y = 0 \qquad \text{(C 1)}$$

for the Landau branch $(n=0)$ of zeros when $k\,\lambda_{\rm D}<0.5$ holds and with

$$x = [\pi\,(n+\tfrac{3}{8})]^{1/2}\,;\quad y = -x \qquad \text{(C 2)}$$

in all other cases.

These initial approximations are sufficient to ensure convergence against the zero of the n-th branch. Because of the steepness and quick oscillation of $Z(\zeta)$ this fact is not obvious. To exclude convergence against the zero of another branch of solutions we control the number of the branch using [cf. Equation (A 5)].

$$n = \frac{1}{2\,\pi}\left(\arctan\frac{y}{x} - 2\,x\,y\right) - \frac{1}{4}\,. \qquad \text{(C 3)}$$

This relation discriminates the branches of zeros even out of the range of validity of Equation (A 5).

D) Time Development of Density Perturbations

The calculation of the time dependent density perturbation by summation of residues is obvious when the loci of the zeros of the dispersion equation are known.

The exact development of the perturbation is evaluated by two different methods:

a) by integration of

$$n(k,t) = \frac{2}{\pi}\,(k\,\lambda_{\rm D})^2\,\beta\int_0^{+\infty}\cos(\sqrt{2}\,\zeta\,k\,u_0\,t)\,\frac{\operatorname{Im}[Z(\beta\,\zeta)\,(k^2\,\lambda_{\rm D}^2+1+\zeta\,Z^*(\zeta))]}{|k^2\,\lambda_{\rm D}^2+1+\zeta\,Z(\zeta)|^2}\,{\rm d}\zeta$$

$$= \frac{2}{\pi}\,(k\,\lambda_{\rm D})^2(1+k^2\,\lambda_{\rm D}^2)\int_0^{\infty}\frac{\cos(\sqrt{2}\,\zeta\,k\,u_0\,t)\operatorname{Im}Z(\zeta)}{|k^2\,\lambda_{\rm D}^2+1+\zeta\,Z(\zeta)|^2}\,{\rm d}\zeta \quad \text{for } \beta=1 \qquad \text{(D 1)}$$

along the real axis.

Difficulties arise for $k\,\lambda_{\rm D} \ll 1$ or $\beta \gg 1$. For $k\,\lambda_{\rm D} \ll 1$ the integrand becomes steep because the Landau zero comes close to the real axis. By subtraction and addition of the contribution of the resonance this difficulty can be remended. For $\beta \gg 1$ the integrand is steep near $\zeta = 0$ due to the nominator $\operatorname{Im} Z(\beta\zeta) = \sqrt{\pi}\exp(-\beta^2\zeta^2)$. This can be overcome by a drastic reduction of the length of the integration step, only.

b) by numerical treatment of the integral equation equivalent to Vlasov's equation

$$n(k,t) = n_f(k,t) - \omega_p{}^2 \int_0^t \tau\, n_F(k,\tau)\, n(k,t-\tau)\,\mathrm{d}\tau \tag{D 2}$$

where n_f and n_F are the free streaming functions

$$n_f = \int f(k,u,t=0)\, e^{-ikut}\,\mathrm{d}u \;;\; n_F = \int F(u)\, e^{-ikut}\,\mathrm{d}u\,. \tag{D 3}$$

Both methods give identical results.

Method b) is free from difficulties when $k\lambda_{\mathrm{D}}$ gets small or β gets large.

Since the time integral (D 2) is of the convolution type the calculation of $n(k,t)$ bases on values of $n(k,t')$ at times $t'<t$, only.

In the dimensionless time variable $\varkappa := \sqrt{2}\,k\,u_0\,t = i\,\delta'$ the recursion reads for equidistant time steps

$$\begin{aligned} n(i) &= n_f(i) - \frac{\delta'^2}{4(k\lambda_{\mathrm{D}})^2}\left[\sum_{j=1}^{i-1} 2\,j\,n_F(j)\,n(i-j) + i\,n_F(i)\right], \\ n(0) &= n_f(0)\,. \end{aligned} \tag{D 4}$$

The main advantage of method b) is that the distribution functions $f(k,u,t=0)$ and $F(u)$ can be choosen arbitrarily whereas in the case a) new subroutines for the Hilbert transforms of these distributions must be developed. So the numerical treatment of non-maxwellian distributions is possible without further preparation.

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1 A. Vlasov, J. Phys. **9**, 25 [1945].
2 L. D. Landau, J. Phys. **10**, 25 [1946].
3 J. F. Denisse, C. R. Acad. Sci. (France) **253**, 1539 [1961]; **254**, 4158 [1962].
4 G. Backus, J. Math. Phys. **1**, 178 [1960].
5 N. J. Hayes, Phys. Fluids **4**, 1387 [1961].
6 H. Weitzner, Proc. of Symp. in Applied Math., p. 127, American Math. Soc., Providence 1967, and references therein.
7 G. Ecker and J. Hölling, Phys. Fluids **6**, 70 [1963].
8 J. D. Jackson, Plasma Phys. (J. Nucl. Energy C) **1**, 171 [1960].
9 J. Dawson, Phys. Fluids **4**, 869 [1961].
10 J. H. Malmberg, C. B. Wharton, and W. E. Drummond, Plasma Phys. and Contr. Nucl. Fus. Res., **I**, p. 485 (1965).
11 J. H. Malmberg, C. B. Wharton, R. W. Gould, and T. M. O'Neill, Phys. Fluids **11**, 1147 [1968].
12 N. G. Van Kampen, Physica **21**, 949 [1955].
13 K. M. Case, Ann. Phys. (N. Y.) **7**, 349 [1959].
14 O. Penrose, Phys. Fluids **3**, 258 [1960].
15 N. G. Van Kampen and B. U. Felderhof, Theoretical Methods in Plasma Physics, North-Holland P.C., Amsterdam 1967.
16 D. B. Chang, Phys. Fluids **7**, 1980 [1964].
17 N. Rostoker, Phys. Fluids **7**, 479 [1964].
18 M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover P., New York 1965.
19 B. D. Fried and S. D. Conte, The Plasma Dispersion Function, Academic Press, New York 1961.